Fields tell matter how to move

Trevor W. Marshall
CCAB, Cardiff University, UK
marshallt@Cardiff.ac.uk

Abstract. Starting from the Oppenheimer-Snyder solution for gravitational collapse, we show by putting it into the harmonic coordinates, for which the distant Riemann metric is galilean, that the final state of collapse for a collapsed star of any mass, including the one thought to occupy the centre of our galaxy, has a finite radius roughly equal to its Schwarzschild radius. By applying an expression for the gravitational energy tensor, we are able to explain the concentration of stellar material in a thin shell close to the surface, which gives an explanation for why such a star does not undergo further collapse to a black hole. The interior of the star is characterized by a low density of the original stellar material, but, far from being empty, this region is occupied by a very high density of gravitational energy; this density is negative and the consequent repulsion is what produces the surface concentration of stellar material.

1 Introduction

There is a widespread belief that space tells matter how to move[1]. This, we believe, has resulted in a profound misunderstanding of gravity. Although the error can be traced back all the way to the founder of the modern theory[2], there is ample evidence that he nevertheless had a strong inclination to go in the direction which we are advocating here.

Throughout the decade in which Einstein discovered what he called General Relativity (GR), he repeatedly attempted to cast gravity within the context of a classical field theory by constructing a tensor for the energy and momentum carried by its field[3]. Perhaps the culmination of this effort was the article[4] in which he derived the quadrupole formula for the emission of gravitational radiation from a bounded nonspherical source. He had already changed his mind twice about such radiation when he wrote that article; throughout the subsequent six decades, not only would he himself change two more times, but his oscillations of opinion would be reflected in the communal understanding of gravity, causing confusion among virtually all of the leading scholars in the field[5]. The communal view changed radically when observation of the Hulse-Taylor pulsar[6] confirmed that this system is losing energy at the rate predicted by Einstein's quadrupole formula.....

But did it? In our opinion the change is incomplete and inadequate. Probably most of us now confidently expect that the LIGO experiment[7] will reveal that gravitational waves passing through space cause a change in the path of light signals exchanged across that same space. The light signals are items of "matter", which is being moved but by what? We have become lazy in repeating, as a kind of mantra, that the moving agent is space itself. That may be an

adequate description of the static solar field's effect on planets and passing light signals, but the time has now come to distinguish between the gravitational field and the space carrying it. The field has energy which may be localized in a determinate manner; the energy has mass which itself gravitates, and above all it must be described by an energy tensor.

In spite of appearing in Einstein's own articles, both before and after GR was conceived[3], the energy "pseudotensor" has not achieved full tensorial status right up to the present day. There is a big obstacle to be overcome, and it was pointed out by Hilbert[9], who was the codiscoverer of the basic Hilbert-Einstein field equation of GR. This is that the metric and its curvature are *the only proper tensors* which may be constructed out of $g_{\mu\nu}$. The analogy with classical field theories suggests that an energy tensor be formed from the derivatives of $g_{\mu\nu}$, but normal derivation is not tensorial and covariant derivation in the Riemann metric gives zero.

It must have been physical intuition which caused Einstein to ignore Hilbert's objection and continue with his "pseudotensor" to deduce the quadrupole formula. In order to make his argument work he made a particular coordinate choice, using what is now called the *harmonic* system, thereby violating the Principle of Equivalence (PE), which he had put at the centre of his derivation of the Hilbert-Einstein equation, and which states that no particular coordinate system is to be preferred. Now we are able to see that, rather than introducing a favoured frame, what he was actually doing was to recognize that the gravitational field is carried by the familiar Minkowski space of so called "Special" relativity. The notion of the gravitational field as being like the electromagnetic field of Faraday and Maxwell was taken up by Einstein's contemporaries de Donder and Lanczos[10][11], and subsequently developed by Fock[12], Rosen[13] and Weinberg[14]. This field interpretation of gravity has subsequently been developed, by Logunov and Mestvirishvili[15], and by Babak and Grishchuk[16], to the point where the Minkowski metric is explicitly present in the field equations.

In advocating the necessity for including the Minkowski metric in the field equations, we will also be stressing the need to distinguish between the various requirements which have been laid on the theories of gravitation, namely covariance, gauge invariance and the Principle of Equivalence. We shall argue for maintaining the first, abandoning of the second, and accepting only the weakest form of the third, in the form of the Eötvös Principle (EoP).

Our new contribution to the field theory of gravity is in gravitational collapse. Starting from the Oppenheimer-Snyder solution[17], we show by putting it into the harmonic coordinates, for which the distant Riemann metric is galilean, that the final state of collapse, for a star of any mass, including the one thought to occupy the centre of our galaxy, has a finite radius roughly equal to its Schwarzschild radius.

2 The energy tensor

We follow Babak and Grishchuk[16] (BG) with some changes of notation. The self-interacting tensor field $h^{\alpha\beta}(x)$ is defined on an underlying (flat) Minkowski space with metric

$$d\sigma^2 = \gamma_{\mu\nu}(x)\, dx_\mu dx_\nu \quad . \tag{1}$$

The d'Alembertian operator is

$$\Box_\gamma = \gamma^{\mu\nu} D_\mu D_\nu \quad , \tag{2}$$

where

$$\gamma^{\mu\nu}\gamma_{\nu\lambda} = \delta^\mu_\lambda \quad , \tag{3}$$

and D_μ denotes covariant differentiation with respect to x_μ in the Minkowski metric. We define

$$\Phi^{\alpha\beta} = \gamma^{\alpha\beta} + h^{\alpha\beta} \quad , \tag{4}$$

and we shall denote covariant differentiation of Φ simply by a lower index, that is

$$\Phi^{\alpha\beta}_\sigma = D_\sigma \Phi^{\alpha\beta} = \gamma^{\alpha\beta}_{;\sigma} + h^{\alpha\beta}_{;\sigma} = h^{\alpha\beta}_{;\sigma} \quad . \tag{5}$$

We define also the inverse field tensor $\Psi_{\alpha\beta}$ by

$$\Psi_{\alpha\beta}\Phi^{\beta\gamma} = \delta^\gamma_\alpha \quad . \tag{6}$$

The field equations are determined from a lagrangian density

$$L = L_g + L_m \quad . \tag{7}$$

The gravitational lagrangian is

$$L_g = -\frac{\sqrt{-\gamma}}{4\kappa}\Phi^{\beta\gamma}_\alpha P^\alpha_{\beta\gamma}, \quad \gamma = \det\left(\gamma_{\mu\nu}\right), \quad \kappa = \frac{8\pi G}{c^4}, \tag{8}$$

where

$$P^\alpha_{\beta\gamma} = \frac{1}{4}\sqrt{\frac{\gamma}{\Phi}}\left[2\Phi^{\sigma\tau}_\tau\left(\delta^\alpha_\beta\Psi_{\gamma\sigma} + \delta^\alpha_\gamma\Psi_{\beta\sigma}\right) + \Phi^{\rho\mu}_\tau\Phi^{\tau\alpha}\left(\Psi_{\beta\gamma}\Psi_{\rho\mu} - 2\Psi_{\beta\rho}\Psi_{\gamma\mu}\right)\right] \quad , \tag{9}$$

and

$$\Phi = \det\left(\Phi^{\alpha\beta}\right) \quad . \tag{10}$$

This is equivalent to an expression first given by Fock (Ref[12], Appendix B). The "matter" lagrangian L_m is a function of all the nongravitational (particle and field) quantities, collectively labelled by ϕ_m, and of $h^{\alpha\beta}$, which enters only through the combination $\sqrt{-\gamma}\Phi^{\alpha\beta}$, that is

$$L_m = L_m\left(\phi_m, \sqrt{-\gamma}\Phi^{\alpha\beta}\right) \quad . \tag{11}$$

The field equations are then obtained by varying L with respect to $h^{\alpha\beta}$, or equivalently $\Phi^{\alpha\beta}$, and are

$$-\frac{2\kappa}{\sqrt{-\gamma}}\frac{\delta L_g}{\delta\Phi^{\mu\nu}} = -D_\alpha P^\alpha_{\mu\nu} - P^\alpha_{\mu\beta}P^\beta_{\nu\alpha} + \frac{1}{3}P^\alpha_{\mu\alpha}P^\beta_{\nu\beta} = \frac{2\kappa}{\sqrt{-\gamma}}\frac{\partial L_m}{\partial\Phi^{\mu\nu}} \quad . \tag{12}$$

BG showed that these are equivalent to the standard Hilbert-Einstein (HE) equations of GR. First one makes the identification of the inverse field tensor with the Riemannian metric tensor, namely

$$g_{\alpha\beta} = \sqrt{\frac{\Phi}{\gamma}}\Psi_{\alpha\beta}, \quad \gamma = \det\left(\gamma^{\alpha\beta}\right) \quad , \tag{13}$$

from which it follows that

$$\det\left(g_{\alpha\beta}\right) = g = \Phi/\gamma^2 \quad . \tag{14}$$

Then, in terms of the new variables, one finds that

$$-\frac{2\kappa}{\sqrt{-\gamma}}\frac{\delta L_g}{\delta\Phi^{\mu\nu}} = R_{\mu\nu} \quad , \tag{15}$$

the right side being the contracted curvature tensor of the Riemannian metric. On the other hand, defining the inverse metric tensor $G^{\mu\nu}$ (conventional notation $g^{\mu\nu}$) by

$$G^{\mu\nu}g_{\nu\lambda} = \delta^\mu_\lambda \quad , \tag{16}$$

and the material stress tensor by

$$T_{\mu\nu} = \frac{2c^2}{\sqrt{-g}}\frac{\partial L_m}{\partial G^{\mu\nu}} \quad , \tag{17}$$

we obtain, since $G^{\mu\nu} = \sqrt{\gamma/\Phi}\Phi^{\mu\nu}$,

$$\frac{2c^2}{\sqrt{-\gamma}}\frac{\partial L_m}{\partial\Phi^{\mu\nu}} = T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}G^{\alpha\beta}T_{\alpha\beta} = T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T \quad . \tag{18}$$

With these substitutions (12) becomes

$$R_{\mu\nu} = \kappa c^2\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \quad , \tag{19}$$

which is the Hilbert-Einstein equation[1].

The energy balance is obtained by varying L with respect to $\gamma_{\mu\nu}$, that is (note that this variation takes account of $\gamma^{\mu\nu}$ occurring in $\Phi^{\mu\nu}$),

$$\frac{\delta L_g}{\delta\gamma_{\mu\nu}} + \frac{\partial L_m}{\partial\gamma_{\mu\nu}} = 0 \quad . \tag{20}$$

[1] Our derivation follows a long line of field theoretic ones, starting with Hilbert, who used the lagrangian $R\sqrt{-g}$. The lagrangian of BG is a covariant version of one used from the years shortly after the establishing of GR (see, for example, Eddington[8] section 58).

The first term is given by

$$-\frac{2}{\sqrt{-\gamma}}\frac{\delta L_g}{\delta\gamma_{\mu\nu}} = -\frac{1}{2\kappa}D_\alpha D_\beta\left(\Phi^{\mu\nu}\Phi^{\alpha\beta} - \Phi^{\mu\alpha}\Phi^{\nu\beta}\right) - \frac{2}{\sqrt{-\gamma}}q^{\alpha\beta\mu\nu}\frac{\delta L_g}{\delta\Phi^{\alpha\beta}} + t_g^{\mu\nu} \quad , \tag{21}$$

where

$$q^{\alpha\beta\mu\nu} = \Phi^{\alpha\mu}\Phi^{\beta\nu} - \frac{1}{2}\Phi^{\alpha\beta}\Phi^{\mu\nu} - \gamma^{\alpha\mu}\gamma^{\beta\nu} + \frac{1}{2}\gamma^{\mu\nu}\Phi^{\alpha\beta} \quad , \tag{22}$$

and[2]

$$\begin{aligned} 16\kappa t_g^{\mu\nu} \;=\; & \left(2\Phi^{\mu\delta}\Phi^{\nu\omega} - \Phi^{\mu\nu}\Phi^{\delta\omega}\right)\left(2\Psi_{\alpha\rho}\Psi_{\beta\sigma} - \Psi_{\alpha\beta}\Psi_{\rho\sigma}\right)\Phi^{\rho\sigma}_{\delta}\Phi^{\alpha\beta}_{\omega} \\ & + 8\Phi^{\rho\sigma}\Psi_{\alpha\beta}\Phi^{\nu\beta}_{\sigma}\Phi^{\mu\alpha}_{\rho} - 8\Phi^{\mu\alpha}\Psi_{\beta\rho}\Phi^{\nu\beta}_{\sigma}\Phi^{\rho\sigma}_{\alpha} - 8\Phi^{\nu\alpha}\Psi_{\beta\rho}\Phi^{\mu\beta}_{\sigma}\Phi^{\rho\sigma}_{\alpha} \\ & + 4\Phi^{\mu\nu}\Psi_{\alpha\rho}\Phi^{\alpha\beta}_{\sigma}\Phi^{\rho\sigma}_{\beta} + 8\Phi^{\mu\nu}_{\rho}\Phi^{\rho\sigma}_{\sigma} - 8\Phi^{\mu\alpha}_{\alpha}\Phi^{\nu\beta}_{\beta} \quad . \end{aligned} \tag{23}$$

Hence, making use of the field equation (12), this first term becomes

$$-\frac{2}{\sqrt{-\gamma}}\frac{\delta L_g}{\delta\gamma_{\mu\nu}} = -\frac{1}{2\kappa}D_\alpha D_\beta\left(\Phi^{\mu\nu}\Phi^{\alpha\beta} - \Phi^{\mu\alpha}\Phi^{\nu\beta}\right) + \frac{2}{\sqrt{-\gamma}}q^{\alpha\beta\mu\nu}\frac{\partial L_m}{\partial\Phi^{\alpha\beta}} + t_g^{\mu\nu} \quad . \tag{24}$$

The second term is given by

$$\frac{\partial L_m}{\partial\gamma_{\mu\nu}} = \left(\gamma^{\mu\alpha}\gamma^{\nu\beta} - \frac{1}{2}\gamma^{\mu\nu}\Phi^{\alpha\beta}\right)\frac{\partial L_m}{\partial\Phi^{\alpha\beta}} \quad . \tag{25}$$

Hence (20) becomes

$$\begin{aligned} \frac{1}{2\kappa}D_\alpha D_\beta\left(\Phi^{\mu\nu}\Phi^{\alpha\beta} - \Phi^{\mu\alpha}\Phi^{\nu\beta}\right) &= t_g^{\mu\nu} - \frac{2}{\sqrt{-\gamma}}\left(\Phi^{\alpha\mu}\Phi^{\beta\nu} - \frac{1}{2}\Phi^{\alpha\beta}\Phi^{\mu\nu}\right)\frac{\partial L_m}{\partial\Phi^{\alpha\beta}} \\ &= t_g^{\mu\nu} + \frac{g}{\gamma}\left(G^{\alpha\mu}G^{\beta\nu} - \frac{1}{2}G^{\alpha\beta}G^{\mu\nu}\right)\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}G^{\sigma\tau}T_{\sigma\tau}\right) \quad . \end{aligned} \tag{26}$$

In the usual notation the inverse Riemann metric is $G^{\alpha\beta} = g^{\alpha\beta}$, and it is the tensor used for raising indices, so that the right side simplifies to give

$$D_\alpha D_\beta\left(\Phi^{\mu\nu}\Phi^{\alpha\beta} - \Phi^{\mu\alpha}\Phi^{\nu\beta}\right) = 2\kappa t^{\mu\nu}, \quad t^{\mu\nu} = \left(t_g^{\mu\nu} + \frac{g}{\gamma}T^{\mu\nu}\right) \quad . \tag{27}$$

Now the expression $D_\alpha D_\beta D_\mu\left(\Phi^{\mu\nu}\Phi^{\alpha\beta} - \Phi^{\mu\alpha}\Phi^{\nu\beta}\right)$ is antisymmetric with respect to a $\mu\alpha$ interchange, from which we may deduce the covariant conservation equation

$$D_\mu t^{\mu\nu} = 0 \quad . \tag{28}$$

The theory summarized in this section has all three basic properties mentioned in the previous section, that is covariance, gauge invariance and weak equivalence as summed up in EoP. The latter may be briefly stated as "All forms of

[2] This expression was given in noncovariant form, that is with ordinary instead of covariant derivatives, by Landau and Lifshitz[18] and also by Fock[12], as the gravitational energy "pseudotensor".

energy gravitate equally". It has an active aspect characterized by the single coupling constant κ in (27), and also a passive aspect, contained in the equation

$$\nabla_\mu T^{\mu\nu} = 0 \quad . \tag{29}$$

Of course, the latter property is a consequence of (19) together with the Bianchi identity

$$\nabla_\mu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) = 0 \quad . \tag{30}$$

3 The harmonic condition

The condition

$$\partial_\mu \Phi^{\mu\nu} = 0 \tag{31}$$

has a history almost as long as GR itself. It was used by Einstein[4] in his derivation of the quadrupole formula, but was almost immediately criticized, for example by Eddington (Ref[8], page 130), for being noncovariant; Eddington said that gravitational waves "travel at the speed of thought", and his criticism resounded throughout the subsequent six decades, finding fame in the title of a seminal history[5] of gravitational waves. In the first decade of GR the harmonic condition was developed by de Donder[10], and also by Einstein's research assistant Lanczos[11]; later it was championed by Fock[12], who claimed that this choice of coordinates was necessary in order to guarantee the condition that there are no ingoing waves at large distance from an isolated massive object. Fock, and also Weinberg[14] have shown that this condition, as well as being an essential part of the field interpretation of GR, is extremely practical in all analyses of the far fields of such a system, for example in establishing post-Newtonian expansions. Subsequently[19] it was argued by Logunov and Mestvirishvili that, in its covariant form

$$D_\mu \Phi^{\mu\nu} = 0 \quad , \tag{32}$$

the harmonic condition, far from being just a choice of coordinate system or "gauge", is, along with the Hilbert-Einstein equation dealt with in the previous section, an essential field equation of gravitation. These authors showed that, by including the Minkowski metric $\gamma_{\mu\nu}$ explicitly in a lagrangian

$$L_{g1} = \lambda \left(2\sqrt{-\gamma}\gamma_{\mu\nu}\Phi^{\mu\nu} - \sqrt{-g} \right) \quad , \tag{33}$$

so that the total lagrangian is

$$L = L_g + L_{g1} + L_m \quad , \tag{34}$$

the covariant harmonic condition does indeed appear as a field equation. Such a modification causes the Hilbert-Einstein equation to acquire two additional terms, one of which is the familiar "cosmological constant" term (what Einstein termed his "biggest mistake", which, however, has now come back into fashion).

The cosmological effect of these two terms has been explored[19] with λ related to the Hubble constant, but in this article we shall assume that λ is negligible, so that the Hilbert-Einstein equation remains unmodified. The extra term $\delta L_{g1}/\delta h^{\mu\nu}$ in the field equation violates the gauge invariance, though not, we stress, the covariance. In general this additional term may result also in the violation of EoP, in the form (29), but the imposition of the field equation (32) is precisely what prevents this occurrence. It should be noted that the loss of gauge invariance arises precisely because of this additional field equation, and that in field theoretic terms it is more accurate to state that what has been lost is the *gauge ambiguity* of GR.

As far as the energy tensor is concerned, we may incorporate the new equation and write the energy balance (27) as

$$t_H^{\mu\nu} = t_{gH}^{\mu\nu} + \frac{g}{\gamma} T^{\mu\nu} \quad , \tag{35}$$

where

$$\begin{aligned} 16\kappa t_{gH}^{\mu\nu} &= \left(2\Phi^{\mu\delta}\Phi^{\nu\omega} - \Phi^{\mu\nu}\Phi^{\delta\omega}\right)\left(2\Psi_{\alpha\rho}\Psi_{\beta\sigma} - \Psi_{\alpha\beta}\Psi_{\rho\sigma}\right)\Phi_\delta^{\rho\sigma}\Phi_\omega^{\alpha\beta} \\ &\quad + 8\Phi^{\rho\sigma}\Psi_{\alpha\beta}\Phi_\sigma^{\nu\beta}\Phi_\rho^{\mu\alpha} - 8\Phi^{\mu\alpha}\Psi_{\beta\rho}\Phi_\sigma^{\nu\beta}\Phi_\alpha^{\rho\sigma} - 8\Phi^{\nu\alpha}\Psi_{\beta\rho}\Phi_\sigma^{\mu\beta}\Phi_\alpha^{\rho\sigma} \\ &\quad + 4\Phi^{\mu\nu}\Psi_{\alpha\rho}\Phi_\sigma^{\alpha\beta}\Phi_\beta^{\rho\sigma} \quad . \end{aligned} \tag{36}$$

Although the two terms on the right side of (35) may be referred to loosely as the "field" and "matter" parts of the energy tensor, it should be remembered that they are not just the functional derivatives of L_g and L_m, as given by (24) and (25); some terms were transferred from one of these to the other. For most purposes it will suffice to consider them both together, and then they may be computed as the left side of (35), that is

$$t_H^{\mu\nu} = \frac{1}{2\kappa} D_\alpha D_\beta \left(\Phi^{\mu\nu}\Phi^{\alpha\beta} - \Phi^{\mu\alpha}\Phi^{\nu\beta}\right) \quad . \tag{37}$$

At this point we specialize to the case of cartesian harmonic coordinates, to be discussed in more detail in the next Section, and for which the Minkowski covariant derivatives are ordinary partial ones. Then the 00-component of $t_H^{\mu\nu}$ reduces to

$$t_H^{00} = \frac{1}{2\kappa} \partial_i \partial_j \left(\Phi^{00}\Phi^{ij} - \Phi^{0i}\Phi^{0j}\right) \quad , \tag{38}$$

which may be expressed as a 3-divergence

$$t_H^{00} = \mathrm{div}\mathbf{t}, \quad t_i = \frac{1}{2\kappa}\partial_j \Theta_{ij}, \quad \Theta_{ij} = \partial_j \left(\Phi^{00}\Phi^{ij} - \Phi^{0i}\Phi^{0j}\right) \quad . \tag{39}$$

A general spherosymmetric field may be written, in cartesian coordinates $x_0 = t, \sqrt{x_1^2 + x_2^2 + x_3^2} = r, x_i/r = n_i$, as

$$\Phi^{00} = \Phi_1(r,t), \quad \Phi^{0i} = \Phi_2(r,t)\, n_i, \quad \Phi^{ij} = \Phi_3(r,t)\, n_i n_j + \Phi_4(r,t)\left(\delta_{ij} - n_i n_j\right) \tag{40}$$

and its determinant is

$$\Phi = \left(\Phi_1\Phi_3 - \Phi_2^2\right)\Phi_4^2 \quad , \tag{41}$$

and so the cartesian tensor on the right side of (39) becomes

$$\Theta_{ij} = \frac{\Phi}{\Phi_4^2} n_i n_j + \Phi_1\Phi_4\left(\delta_{ij} - n_i n_j\right) \quad , \tag{42}$$

that is

$$\mathbf{t} = \frac{\mathbf{n}}{2\kappa}\left[\partial_r \frac{\Phi}{\Phi_4^2} + \frac{2}{r}\left(\frac{\Phi}{\Phi_4^2} - \Phi_1\Phi_4\right)\right] \quad . \tag{43}$$

As a first application we use t_H^{00} to find the field energy distribution in the exterior of a nonstatic spherosymmetric $T^{\mu\nu}$ with total mass M. By Birkhoff's theorem the field is static, and is the harmonic version of the Schwarzschild metric, that is (see Ref[12] 209-215) putting $m = GM/c^2$

$$\Phi_1 = \frac{(r+m)^3}{r^2\,(r-m)}, \quad \Phi_2 = 0, \quad \Phi_3 = -\frac{r^2 - m^2}{r^2}, \quad \Phi_4 = -1, \quad \Phi = -\left(\frac{r+m}{r}\right)^4 \quad , \tag{44}$$

it follows that

$$\mathbf{t} = \frac{Mc^2\mathbf{n}}{4\pi r^2}\left(\frac{r+m}{r}\right)^3 \frac{2r-m}{2r-2m} \quad . \tag{45}$$

We may define a new vector $\mathbf{t}_1$ by subtracting a divergence-free vector

$$\mathbf{t}_1 = \mathbf{t} - \frac{Mc^2\mathbf{n}}{4\pi r^2} \quad , \tag{46}$$

and then the total (matter plus field) energy contained in $r > r_1$ is $Mc^2\mu\,(r_1)$, where

$$\mu\,(r_1) = \int_{r>r_1} \mathrm{div}\mathbf{t}_1 d^3\mathbf{r} = 1 - \left(\frac{r_1+m}{r_1}\right)^3 \frac{2r_1 - m}{2r_1 - 2m} \quad , \tag{47}$$

but in this case it is entirely field energy. Note that this quantity is negative; if $r_1 \gg m$ it is

$$\int_{r_1}^{\infty} t_H^{00} d^3\mathbf{r} = -\frac{7}{2}Mc^2\left[\frac{m}{r_1} + O\left(\frac{m^2}{r_1^2}\right)\right] \quad , \tag{48}$$

and, for example, the absolute value of the field energy in the exterior space of our Sun, for which $m/r_1 \approx 2\times 10^{-6}$, is about three times the mass of our planet.

More generally, we now show that, in the spherosymmetric case, there is a simple connection between the cartesian tensor Θ_{ij} and the spatial components g_{ij}, of the Riemannian metric of GR. In terms of the components of the field tensor we obtained (see equation (42))

$$\Theta_{ij} = \frac{\Phi}{\Phi_4^2} n_i n_j + \Phi_1\Phi_4\left(\delta_{ij} - n_i n_j\right) \quad . \tag{49}$$

The inverse field tensor is

$$\Psi_{00} = \frac{\Phi_3 \Phi_4^2}{\Phi}, \quad \Psi_{0i} = -\frac{\Phi_2 \Phi_4^2}{\Phi} n_i, \quad \Psi_{ij} = \frac{\Phi_1 \Phi_4^2}{\Phi} n_i n_j + \frac{1}{\Phi_4} \left(\delta_{ij} - n_i n_j\right) \tag{50}$$

Since in these coordinates $\gamma = -1$, the corresponding Riemannian metric tensor, from (13), is

$$g_{\alpha\beta} = \sqrt{-\Phi}\,\Psi_{\alpha\beta} \quad , \tag{51}$$

that is

$$g_{00} = \frac{\Phi_3 \Phi_4^2}{\sqrt{-\Phi}}, \quad g_{0i} = -\frac{\Phi_2 \Phi_4^2}{\sqrt{-\Phi}} n_i, \quad g_{ij} = \frac{\Phi_1 \Phi_4^2}{\sqrt{-\Phi}} n_i n_j + \frac{\sqrt{-\Phi}}{\Phi_4} \left(\delta_{ij} - n_i n_j\right) \quad . \tag{52}$$

Converting to spherical polar coordinates we have the metric

$$ds^2 = g_{00} dt^2 + 2 g_{0r} dt dr + g_{rr} dr^2 + g_{\theta\theta} \left(d\theta^2 + \sin^2\theta d\phi^2\right) \quad , \tag{53}$$

where

$$g_{0r} = -\frac{\Phi_2 \Phi_4^2}{\sqrt{-\Phi}}, \quad g_{rr} = \frac{\Phi_1 \Phi_4^2}{\sqrt{-\Phi}}, \quad g_{\theta\theta} = r^2 \frac{\sqrt{-\Phi}}{\Phi_4} \quad . \tag{54}$$

Then (42) becomes

$$\Theta_{ij} = -\frac{g_{\theta\theta}^2}{r^4} n_i n_j - \frac{g_{\theta\theta} g_{rr}}{r^2} \left(\delta_{ij} - n_i n_j\right) \quad , \tag{55}$$

which gives

$$t_H^{00} = \mathrm{div}\mathbf{t}, \quad \mathbf{t} = -\frac{\mathbf{n}}{\kappa} \left(\frac{\partial g_{\theta\theta}}{\partial r} + \frac{g_{\theta\theta} - g_{rr}}{r}\right) g_{\theta\theta} \quad , \tag{56}$$

and hence the total energy (matter plus field) in $r > r_1$ at time t is $\mu\left(r_1, t\right) M c^2$, where

$$1 - \mu\left(r, t\right) = \frac{g_{\theta\theta}}{2 m r^3} \left(g_{\theta\theta} + r^2 g_{rr} - r \frac{\partial g_{\theta\theta}}{\partial r}\right) \quad . \tag{57}$$

4 The Oppenheimer-Snyder solution

We now extend the latter calculation to a continuous mass distribution in order to discuss the gravitational collapse of a spherical star. In contrast with the negative energy distribution of the field, the mass tensor[3] $T^{\mu\nu}$ has the positive-energy property that its contraction $T^{\mu\nu}\eta_\mu\eta_\nu$ with any timelike covariant vector η_μ is positive. One of the few exact solutions for a Riemannian metric derived

[3] We use the short name, favoured by Fock, for this tensor. Neither it nor the more usual "material stress tensor" are quite exact according to our present perspective, because the gravitational field is material and has both energy and mass.

from such a mass tensor is that of Oppenheimer and Snyder[17] (OS) derived in a comoving and synchronous coordinate system, for which the only nonzero component of $T^{\mu\nu}$ is a positive T^{00},

$$ds^2 = d\tau^2 - \left\{\partial_R^* W\right\}^2 dR^2 - W^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \quad , \tag{58}$$

where the operator ∂_R^* denotes differentiation with respect to R at constant τ, and

$$W = u\left[\sqrt{\frac{R^3}{u^3}} - \frac{3\tau\sqrt{2m}}{2}\right]^{2/3} \quad . \tag{59}$$

The function $u\,(R)$ gives the stellar material density, and is a positive monotone increasing function for $R > 0$ with $u\,(0) = 0$ and $u\,(\infty) = 1$. We now choose units such that $2m = 1$ and define a new coordinate

$$v = \sqrt{\frac{W}{u}} = \left[\sqrt{\frac{R^3}{u^3}} - \frac{3\tau}{2}\right]^{1/3} \quad , \tag{60}$$

to replace τ, so that, denoting differentiation at constant v by ∂_R ,

$$\partial_\tau = -\frac{1}{2v^2}\partial_v, \quad \partial_R^* = \partial_R + \frac{\beta}{2v^2}\partial_v, \quad \beta\,(R) = \left(\frac{R}{u}\right)^{3/2}\left(\frac{1}{R} - \frac{u'}{u}\right) \quad , \tag{61}$$

and

$$\partial_R^* W = \xi v^2, \quad \xi = u' + \frac{\beta u}{2v^3} \quad . \tag{62}$$

Then the Riemannian interval in terms of (R, v, θ, ϕ) is

$$ds^2 = \left(\beta dR - 2v^2 dv\right)^2 - \xi^2 v^4 dR^2 - u^2 v^4\left(d\theta^2 + \sin^2\theta d\phi^2\right) \quad , \tag{63}$$

that is

$$\begin{aligned} g_{vv} &= 4v^4, \quad g_{vR} = -2\beta v^2, \quad g_{RR} = \beta^2 - \xi^2 v^4 \quad , \\ g_{\theta\theta} &= \csc^2\theta g_{\phi\phi} = -u^2 v^4 \quad . \end{aligned} \tag{64}$$

The contravariant inverse of this (we are now reverting to conventional notation) is,

$$\begin{aligned} g^{RR} &= \frac{1}{4v^4} - \frac{\beta^2}{4\xi^2 v^8}, \quad g^{vR} = -\frac{\beta}{2\xi^2 v^6}, \quad g^{vv} = -\frac{1}{\xi^2 v^4} \quad , \\ g^{\theta\theta} &= \sin^2\theta g^{\phi\phi} = -\frac{1}{u^2 v^4} \quad . \end{aligned} \tag{65}$$

The Cartesian harmonic coordinates x^μ are obtained, (see Ref[14] pp165-168), as solutions of the equations

$$\Box_g x^\mu = 0 \quad (\mu = 0, 1, 2, 3) \quad , \tag{66}$$

where $\Box_g$ is the Riemannian (not to be confused with the Minkowskian (2)) d'Alembertian

$$\Box_g = \frac{1}{\sqrt{-g}}\partial_\alpha \left(g^{\alpha\beta}\sqrt{-g}\partial_\beta\right) \quad . \tag{67}$$

In spherical coordinates ($t = x_0, r = \sqrt{x_1^2 + x_2^2 + x_3^2}$) these reduce to two equations for what OS term the "exterior time" and "exterior radius", namely

$$Qt = 0, \quad \left(Q - \frac{2\xi^2}{u^2}\right) r = 0 \quad , \tag{68}$$

where $Q = -\xi^2 v^4 \Box_g$ and is given by

$$Q = \frac{\xi}{u^2v^2}\left(\partial_R + \frac{\beta}{2v^2}\partial_v\right)\frac{u^2v^2}{\xi}\left(\partial_R + \frac{\beta}{2v^2}\partial_v\right) - \frac{\xi}{4v^4}\partial_v v^4 \xi \partial_v \quad . \tag{69}$$

However, in the present treatment the coordinates (t, r) describe both the exterior and the interior of the star.

The free-space limit of the last Section corresponds to $u = 1, \beta = \sqrt{R}, \xi = \sqrt{R}/v^3$, for which the solutions of the latter equations are[15]

$$r = v^2 - \frac{1}{2}, \quad t = \frac{2}{3}R^{3/2} - 2\Psi(v), \quad \Psi(v) = \int_{w_0}^{v} \frac{w^4 dw}{w^2 - 1} \quad (w_0 > 1) \quad , \tag{70}$$

and, substituted in the OS metric (63), these give the harmonic version of the Schwarzschild metric, that is

$$ds^2 = \frac{2r-1}{2r+1}dt^2 - \frac{2r+1}{2r-1}dr^2 - \left(r + \frac{1}{2}\right)^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \quad , \tag{71}$$

and this corresponds to the far field $\Phi^{\mu\nu}$ we used in the previous Section. Both of the PDEs (68) have a singularity along the curve $v = v_0(R)$, specified by

$$\frac{dv_0}{dR} = \frac{1}{2}u' - \frac{\beta}{2v_0^3}(v_0 - u), \quad v_0(\infty) = 1 \quad . \tag{72}$$

In our earlier article[20], we showed, for a particular choice of u, how to solve (68) numerically by integrating along a family of characteristics, satisfying the ordinary differential equation

$$\frac{dv}{dR} = \frac{1}{2}u' - \frac{\beta}{2v^3}(v - u) \quad . \tag{73}$$

The operation of differentiation along such a characteristic is

$$\partial_R^C = \partial_R - \left(\frac{\xi}{2} - \frac{\beta}{2v^2}\right)\partial_v \quad , \tag{74}$$

and the operator Q may be expressed as

$$Q = \left(\partial_R^C\right)^2 + \xi\partial_v\partial_R^C + q\partial_R^C + \xi^2\left(\frac{1}{u} - \frac{1}{v}\right)\partial_v \quad , \tag{75}$$

where

$$q = \frac{2u'}{u} + \frac{u'}{\xi} + \frac{u}{\xi}\left(\frac{5\beta^2}{2v^6} - \frac{\beta'}{v^3}\right) \quad . \tag{76}$$

We now define the *physical region* as $R > 1, v > v_0(R)$; it is our contention that (68) describes the whole evolution of the collapsing system, without any singularity, in this region, and the final stage of the collapse, corresponding to the limit $t(R,v) \to +\infty$, is described by the values of r and t close to the limit characteristic $v = v_0(R)$. A necessary preliminary to a numerical study of these equations is the obtaining of asymptotic expansions, both for large R and for large v when R is small, and this was done in our earlier article[20]. An extension of this calculation may now be made by applying these earlier results to the total energy distribution given by (57), putting

$$g_{\theta\theta} = -u^2 v^4 \quad , \tag{77}$$

and

$$\begin{aligned} g_{rr} &= \left(2v^2\frac{\partial v}{\partial r} - \beta\frac{\partial R}{\partial r}\right)^2 - \xi^2 v^4 \left(\frac{\partial R}{\partial r}\right)^2 \\ &= \frac{4v^4}{J^2}\frac{\partial^C t}{\partial R}\left(\frac{\partial^C t}{\partial R} + \xi\frac{\partial t}{\partial v}\right) \quad , \end{aligned} \tag{78}$$

where

$$J = \frac{\partial^C t}{\partial R}\frac{\partial r}{\partial v} - \frac{\partial^C r}{\partial R}\frac{\partial t}{\partial v} \quad , \tag{79}$$

to give

$$1-\mu(R,v) = \frac{u^4 v^8}{2mr^3}\left[1 - \frac{2r}{J}\left\{\frac{2}{v}\frac{\partial^C t}{\partial R} - \xi\left(\frac{1}{u} - \frac{1}{v}\right)\frac{\partial t}{\partial v}\right\} - \frac{4r^2}{J^2u^2}\frac{\partial^C t}{\partial R}\left(\frac{\partial^C t}{\partial R} + \xi\frac{\partial t}{\partial v}\right)\right] \tag{80}$$

.

It is appropriate to mention again here that $t_H^{\mu\nu}(r,t)$, from which the quantity $\mu(R,v)$ was derived, is indeed a tensor. If, for example, we should choose to transform it back to the original OS coordinates, this would be possible, provided we bear in mind that in those coordinates the partial derivatives ∂_α and ∂_β would be replaced by the Minkowski covariant derivatives D_α and D_β; the latter, of course, are not the same as the Riemann covariant derivatives ∇_α and ∇_β. In order to describe the actual energy distribution we need μ as a function of (r,t), which requires rather complicated interpolations from $\mu(R,v), r(R,v)$ and $t(R,v)$, and we shall describe such a detailed calculation in another article. Nevertheless we venture to draw some qualitative conclusions from the results already obtained.

5 What we expect to find

Figure 1: The energy content $1-\mu(r)$, in units of Mc^2, contained within a sphere of radius r measured from the centre of a collapsing star, as a function of r for fixed t. Between A and B the energy is mainly gravitational and negative, between B and C it is mainly stellar material and positive, and beyond C it is again gravitational and negative. The radius r is in units of the Schwarzschild radius.

The field part of the 00-component of the energy tensor is negative, as we found in an earlier section when we examined the far field. In the deep interior of the star we expect to find that this component is substantial, and indeed bigger in absolute magnitude than T^{00}. Being negative, it produces, in a bootstrapping manner, its own gravitational field, *and this is repulsive.* That explains why, as we already found in our earlier article[20], the material energy T^{00} is concentrated in a shell near the surface as $t \to +\infty$. The density function $u(R)$ describes a star with an initially diffuse corona, but the formation of a distinct surface is an expected part of its evolution. As pointed out above, a substantial amount of computation is required in order to obtain a quantitative expression for the total (gravitational energy plus stellar material) mass distribution, $\mu(r,t)$. In Figure 1 we give an artist's impression of this profile, for fixed t, at a fairly advanced stage of the collapse process, when the surface shell has begun to form.

The key element in our approach, stressed also in our earlier article[20], is that the coordinate-free topological analysis, now common in GR, fails to ac-

knowledge the central role of the energy tensor. We accept the Hilbert-Einstein field equations, but add a supplementary set which results in the inertial (that is harmonic) coordinate system being privileged. That means maintaining both covariance and a weak principle of equivalence, in the form of EoP, but rejecting gauge invariance. As was emphasized first by Fock, this latter "loss" means taking the "G" out of GR, leaving us, nevertheless, with a theory which Fock himself called Einstein's Theory of Gravitation.

As far as the currently fashionable notion of "black holes" is concerned, our conclusion is that there is a limit to the degree of compactification undergone in a collapsing gravitational system, and this limit is set by the properties of the gravitational field itself rather than the presence of the other forces of nature, like neutron degenerative pressure. Relatively small "condensars" are observed as neutron stars, or pulsars, but heavier ones, like the one thought to lie at the centre of our galaxy, are more diffuse, probably of white-dwarf density. Our analysis is of an idealized, purely gravitational system, and our key finding is the limit characteristic curve, which represents the limit $t \to +\infty$. This curve is equivalent to the "event horizon" popularly acknowledged to come out of coordinate-free GR, but for us there is *nothing* the other side of the limit characteristic. The other side of our "event horizon" lies outside the physical space; to go there is to go "to infinity and beyond" along with maybe Buzz Lightyear or Doctor Who!

References

[1] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation,* (Freeman, San Francisco, 1973)

[2] A. Einstein, *Sitzungsber. preuss. Akad. Wiss.* **48,** 844-847 (1915)

[3] A. Einstein, *Naturforsch. Gesellschaft Zurich* **58,** 284-290 (1913)

[4] A. Einstein, *Sitzungsber. preuss. Akad. Wiss.* **1,** 154-167 (1918)

[5] D. Kennefick,*Traveling at the Speed of Thought* (Princeton Univ Press, 2007)

[6] R. A. Hulse and J. H. Taylor, *Astrophys. J.* **195,** L51-53 (1975)

[7] *Report on the LIGO project,* http://www.ligo-la.caltech.edu/LLO/overviewsci.htm

[8] A. S. Eddington, *The Mathematical Theory of Relativity,* (Cambridge Univ Press,1923)

[9] D. Hilbert, *Goettinger Nachrichten* **4**,21 (1917)

[10] T. de Donder, *La gravitique einsteinienne*, (Gauthier Villars, Paris, 1921)

[11] C. Lanczos, *Phys. Z.* **23**, 537 (1923)

[12] V. A. Fock, *Space, Time and Gravitation,* (Pergamon, Oxford,1966)

[13] N. Rosen, *Phys. Rev.* **57,** 147-153 (1940)

[14] S. Weinberg, *Gravitation and Cosmology* (John Wiley, New York, 1972)

[15] A. A. Logunov and M. A. Mestvirishvili, *The Relativistic Theory of Gravitation* (Mir, Moscow, 1989)

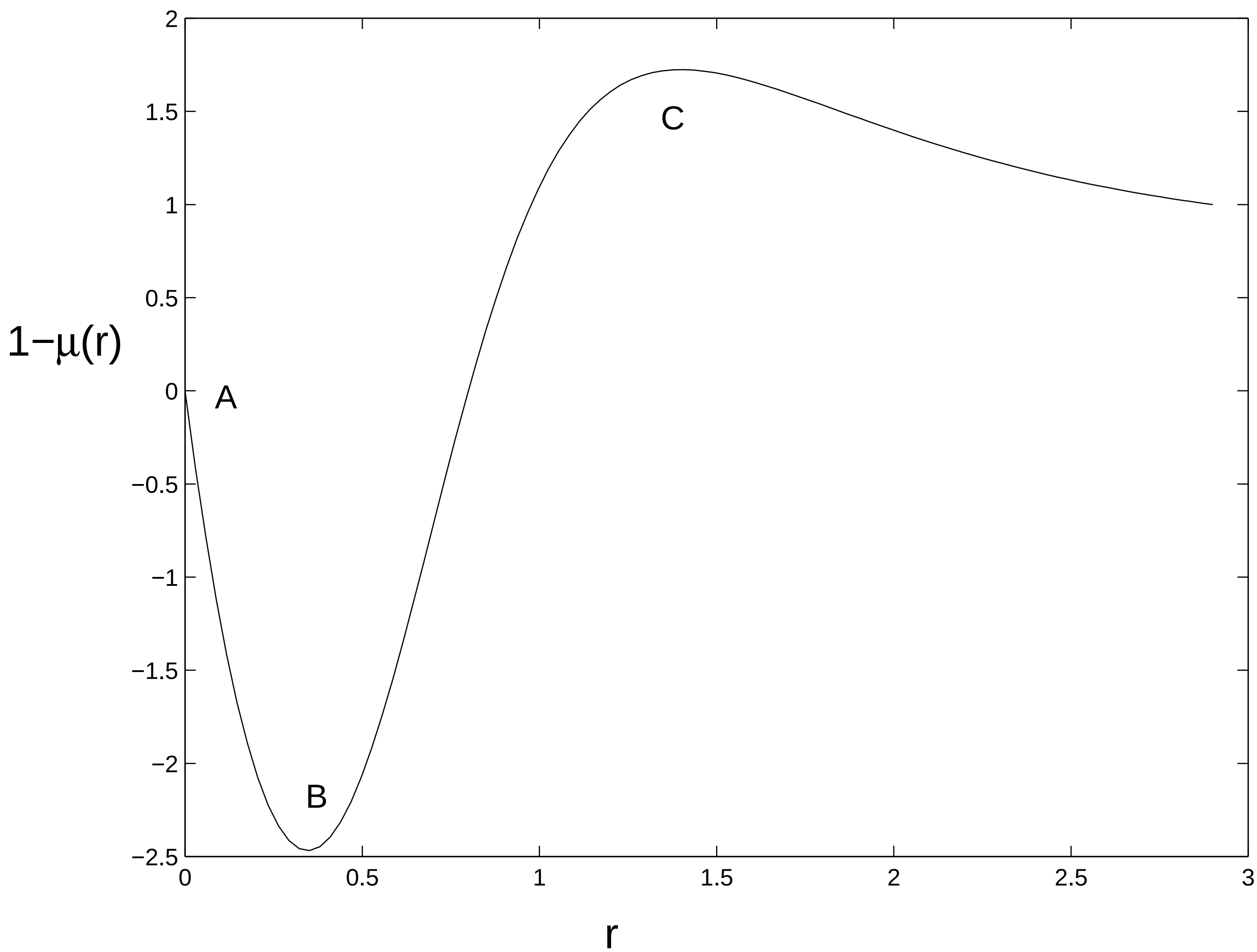
[16] S. V. Babak and L. P. Grishchuk, *Phys. Rev. D* **61,** 024038 (1999)

[17] J. R. Oppenheimer and H. Snyder, *Phys.Rev.* **56,** 455-459 (1939)

[18] L. D. Landau and E. M. Lifshitz, *Classical Theory of Fields* (Addison Wesley, Cambridge, Mass, 1951)

[19] A. A. Logunov, *Theory of Gravity* (Nauka, Moscow, 2001)

[20] T. W. Marshall and M. K. Wallis *J. Cosmology,* **6,** 1473-1484 (2010)


1−μ(r)
r
A
B
C
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
0
0.5
1
1.5
2
2.5
3